Exercise 32

(a) Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} + xy\,\mathbf{j}$ on a particle that moves once around the circle $x^2 + y^2 = 4$ oriented in the counterclockwise direction.

(b) Use a computer algebra system to graph the force field and circle on the same screen. Use the graph to explain your answer to part (a).

Solution

Begin by parameterizing the particle's path on the circle: $x(t) = 2\cos t$ and $y(t) = 2\sin t$ with $0 \le t \le 2\pi$. With this parameterization in t, the work done by the force field is

$$\begin{aligned}
W &= \int_C \mathbf{F} \cdot d\mathbf{r} \\
&= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \\
&= \int_0^{2\pi} \langle [x(t)]^2, x(t)y(t) \rangle \cdot \frac{d}{dt}\langle x(t), y(t) \rangle\,dt \\
&= \int_0^{2\pi} \langle (2\cos t)^2, (2\cos t)(2\sin t) \rangle \cdot \frac{d}{dt}\langle 2\cos t, 2\sin t \rangle\,dt \\
&= \int_0^{2\pi} \langle 4\cos^2 t, 4\cos t \sin t \rangle \cdot \langle -2\sin t, 2\cos t \rangle\,dt \\
&= \int_0^{2\pi} [(4\cos^2 t)(-2\sin t) + (4\cos t \sin t)(2\cos t)]\,dt \\
&= \int_0^{2\pi} (0)\,dt \\
&= 0.
\end{aligned}$$

Looking at the vector field and the path traversed, the work is zero because the force vectors are perpendicular to the particle's path at every point.